Resonant activation in a system with deterministic oscillations of barrier height

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A thermally relaxing system with a harmonically oscillating barrier height is considered. The dynamics of the system are described by a Smoluchowski equation with a time dependent right hand side. For both absorbing and reflecting boundary conditions, the solutions of this equation show that the oscillating system has the same resonant properties, and the same dependence on initial conditions, respectively, on the phase of the harmonic oscillations, as a conventional resonant system in which the barrier executes dichotomic Markovian fluctuations.

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I. INTRODUCTION

The phenomenon of resonant activation is generally associated with thermally relaxing systems in which the confining barrier executes random fluctuations [1–4]. We find, however, that the barrier fluctuations need not be random, and show here that resonant activation is found also in systems performing deterministic harmonic oscillations of frequency γ. Moreover, depending on the phase of the oscillations time $t=0$, the relaxation time may, in this case, have a single minimum as a function of γ, or a minimum and a maximum, or no extremum at all.

II. DETERMINISTIC OSCILLATIONS

We consider an overdamped system described in scaled units by the Smoluchowski equation

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[\frac{dV}{dx}+\frac{\partial}{\partial x}\right]P. \tag{1}$$

For a general time dependent potential $V=V(x,t)$ this equation may be too difficult to solve with satisfactory precision, and for this reason we confine ourselves to the special case

$$V=V(x,t|\phi)=2V_a\sin\{2\pi(\gamma t+\phi)\}\sin^2 x, \tag{2}$$

where γ is the frequency of the harmonic oscillations of the potential V, and ϕ is their phase. Imposing further on Eq. (1) the absorbing boundary conditions $P(\pm\pi/2,t)=0$, we seek the nonequilibrium distribution function $P(x,t)$, $P(x,0)=\delta(x)$, in the form

$$P(x,t)=\sum_{k=0}^{\infty}a_k(t)\cos(2k+1)x. \tag{3}$$

The decay law then becomes

$$W(t)=\int_{-\pi/2}^{\pi/2}dx P(x,t)=2\sum_{k=0}^{\infty}(-1)^k\frac{a_k(t)}{2k+1}, \tag{4}$$

with $W(0)=1$ and $a_k(0)=2/\pi$ by virtue of the initial conditions, and the relaxation time τ of the decaying system is defined by the integral

$$\tau(\gamma,\phi)=\int_0^{\infty}dt W(t). \tag{5}$$

Upon substituting Eq. (3) into the Smoluchowski Eq. (1) we find that the coefficients $a_k(t)$ satisfy the tridiagonal system of linear differential equations

$$\dot{a}_0=-(1-v)a_0-va_1, \tag{6}$$

$$\dot{a}_k=(2k+1)va_{k-1}-(2k+1)^2a_k-(2k+1)va_{k+1}, \tag{7}$$

$k\geqslant 1$, where $\dot{a}_k=da_k/dt$, and

$$v=v(t)=V_a\sin\{2\pi(\gamma t+\phi)\}$$

for brevity. We apply to this differential system the backward Euler method [5], and solve then (after cutoff) the resultant tridiagonal linear system using Lorentz unit (LU) decomposition [6]. The accuracy of the computation is verified by varying the cutoff value (typically $N\lesssim 5000$ is sufficient) and the time step of the backward Euler integration.

The behavior of the relaxation time $\tau(\gamma,\phi)$ of Eq. (5) is apparent from Fig. 1 and from the more detailed selected plots of Fig. 2. According to these figures, the relaxation time has a resonant minimum as a function of γ for almost all values of the phase ϕ, except for those close to $\phi=3/4$.

Particularly interesting is the behavior of the function $\tau(\gamma,0)$ shown in Fig. 2: There is $V(x,t|0)|_{\gamma=0}\equiv 0$, and accordingly $\tau(0,0)=\tau_{\mathrm{free}}=\pi^2/8$ corresponds to the free particle result. With increasing frequencies the relaxation time initially increases [as the energy barrier to be overcome is greater than zero for $t\in(0,1/2\gamma)$] towards a local maximum. At even greater frequencies the function $\tau(\gamma,0)$ then exhibits a resonant minimum, and $\lim_{\gamma\to\infty}\tau(\gamma,0)=\tau_{\mathrm{free}}$ in accordance with Ref. [2]. With increasing temperature (decreasing reduced energy V_a) the local maximum shifts to the right and the local minimum to the left along the γ axis. At the same time the resonance amplitude decreases, and the free particle result is recovered in the limit $V_a\to 0$.

At $\phi=1/4$ and $\phi=1/2$ the barrier height initially decreases at $t\geqslant 0$, and the curves $\tau(\gamma,1/4)$ and $\tau(\gamma,1/2)$ therefore exhibit the customary single resonant extremum (minimum), though with a lesser resonant amplitude for $\phi=1/2$. The function $\tau(\gamma,3/4)$, on the other hand, monotonically in-

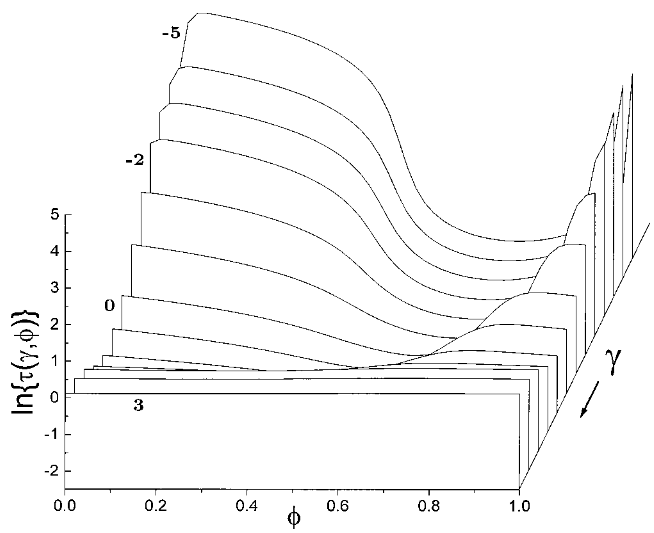

FIG. 1. The relaxation time $\tau = \tau(\gamma,\phi)$ versus the phase ϕ at selected values of the harmonic oscillations frequency γ, with $\ln\gamma = -5$ (labeled), -4, -3.5, -3, -2 (labeled), -1, 0 (labeled), 0.5, 1, 1.5, 2, 2.5, and 3 (frontmost curve). Absorbing boundary conditions, and $V_a = 10$, so that the reduced barrier height is in the interval $\langle -20,20 \rangle$. See Fig. 2 for plots of $\tau(\gamma,\phi)$ versus γ at selected values of ϕ.

creases with γ, since at $t=0$ the barrier is in the inverted configuration and the relaxation rate $\tau \gg 1/\gamma$ for all but the highest driving frequencies.

We have also computed an average over a random phase ϕ,

$$\tilde{\tau}(\gamma) = \int_0^1 d\phi \quad \tau(\gamma,\phi), \tag{8}$$

and found it to be dominated at small γ by the large τ values close to $\phi = 1/4$ (see Figs. 1 and 2), and to have a single extremum, viz the resonant minimum.

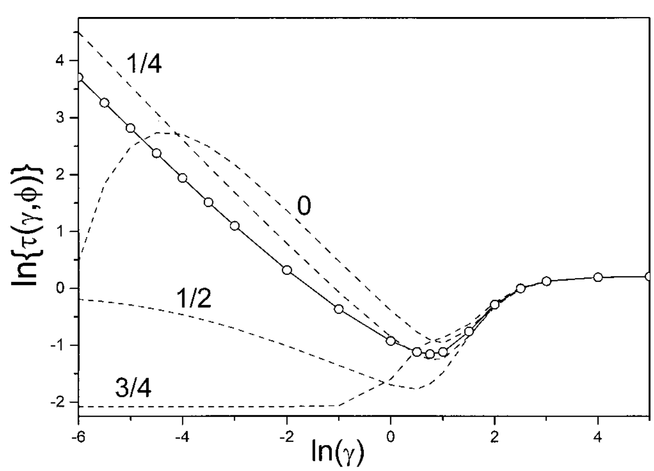

FIG. 2. The relaxation time $\tau = \tau(\gamma,\phi)$ versus the harmonic oscillations frequency γ at selected values of the phase $\phi = 0$, 1/4, 1/2, and 3/4 (labeled dashed lines). Also shown is an average over a uniform distribution of ϕ (solid, ○-marked line). Absorbing boundary conditions, and $V_a = 10$ as in Fig. 1. Compare the average and $\phi = 3/4$ curves of $\tau = \tau(\gamma)$ shown here with the plots of Fig. 3.

III. RANDOM FLUCTUATIONS

The results of the preceding section suggest two conclusions: First, that resonant activation does exist in systems executing deterministic oscillations, and second, that the nature of the resonance effect in this case depends strongly on the configuration of the system at $t=0$, i.e., on the initial conditions. In order to further investigate this dependence we have also analyzed the dynamics of a system that executes dichotomic Markovian fluctuations of rate γ [1]. The evolution operator in this case is

$$\frac{\partial}{\partial t}\begin{pmatrix} P_+ \\ P_- \end{pmatrix} = \begin{pmatrix} L_+ - \gamma & \gamma \\ \gamma & L_- - \gamma \end{pmatrix}\begin{pmatrix} P_+ \\ P_- \end{pmatrix}, \tag{9}$$

where the Smoluchowski operators

$$L_\pm = \frac{\partial}{\partial x}\left[\frac{dV}{dx} \pm \frac{\partial}{\partial x}\right]. \tag{10}$$

We set $V(x) = 2V_a \sin^2 x$, expand the distribution functions $P_\pm(x,t)$ according to Eq. (3), and solve again the resultant pentdiagonal linear differential system,

$$\dot{a}_0^{(\pm)} = -(1+\gamma \mp V_a)a_0^{(\pm)} + \gamma a_0^{(\mp)} \mp V_a a_1^{(\pm)}, \tag{11}$$

$$\dot{a}_k^{(\pm)} = \pm(2k+1)V_a a_{k-1}^{(\pm)} - [\gamma + (2k+1)^2]a_k^{(\pm)} + \gamma a_k^{(\mp)} \mp (2k+1)V_a a_{k+1}^{(\pm)}, \tag{12}$$

$k \geq 1$, by the backward Euler method [5] and subsequent LU decomposition [6].

The resultant relaxation time

$$\tau(\gamma) = \int_0^\infty dt \quad \int_{-\pi/2}^{\pi/2} dx[P_+(x,t) + P_-(x,t)] \tag{13}$$

is plotted in Fig. 3 for two sets of initial conditions: In the first set, with $P_\pm(x,0) = \delta(x)/2$ [1], the system finds itself with equal probability in the up and the down configurations at $t=0$, while the second set, $P_+(x,0) = 0$ and $P_-(x,0) = \delta(x)$, so that at $t=0$ the system is with certainity inverted (down) potential configuration. The computed functions $\tau(\gamma)$ are obviously exact counterparts of the curves $\tilde{\tau}(\gamma)$ and $\tau(\gamma,3/4)$ discussed in the previous section. It should also be noted that the amplitude of the potential fluctuations is here effectively larger than in the case of the continuous deterministic oscillations, and therefore, the resonant amplitude is larger as well, and the resonant minimum is shifted to higher values of γ. The two level system has no analogue of the $\tau(\gamma,0)$ curve shown in Fig. 2, but it is probable that a similar type of behavior could be found in a discrete multilevel system.

IV. A CONCLUDING REMARK

We have examined here two models of a thermally relaxing system with time dependent barrier height. In the first

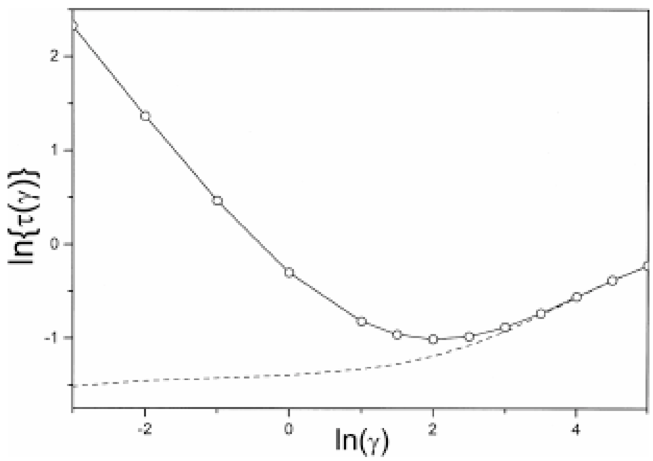

FIG. 3. The relaxation time $\tau=\tau(\gamma)$ versus the frequency γ of dichotomic Markovian fluctuations. The dashed line corresponds to the initial conditions $P_1(x,0)=0$ and $P_2(x,0)=\delta(x)$ [barrier upside down, see Eq. (9) in text], whereas the solid, ○-marked line represents an average over a uniform distribution of initial values, with $P_1(x,0)=P_2(x,0)=\delta(x)/2$. Absorbing boundary conditions, $V_a=10$ as in Figs. 1 and 2.

model, the barrier executes deterministic harmonic oscillations, and in the second model dichotomic Markovian fluctuations. The relaxation properties of the two systems were found to be qualitatively identical, with both systems exhibiting a resonance effect and a similar dependence on initial conditions. In conclusion we wish to show that these properties are shared also by systems in which only intrawell thermal relaxation takes place.

To this end we revert to the Smoluchowski Eq. (1), impose on it the reflecting boundary conditions $\partial P(\pm\pi/2,t)/\partial x=0$, and expand the distribution function $P(x,t)$ as

$$P(x,t)=\sum_{k=0}^{\infty} b_k(t)\cos 2kx, \tag{14}$$

with $P(x,0)=\delta(x)$ as before. The system is alternately monostable and bistable, but $P(x,t)=P(-x,t)$ by symmetry, and no net flux over the barrier takes place. The system is also probability conserving, and a relaxation time must be defined with respect to a dynamic variable. We select here, for this purpose, the expectation value of $\cos 2x$, and define the relaxation time as

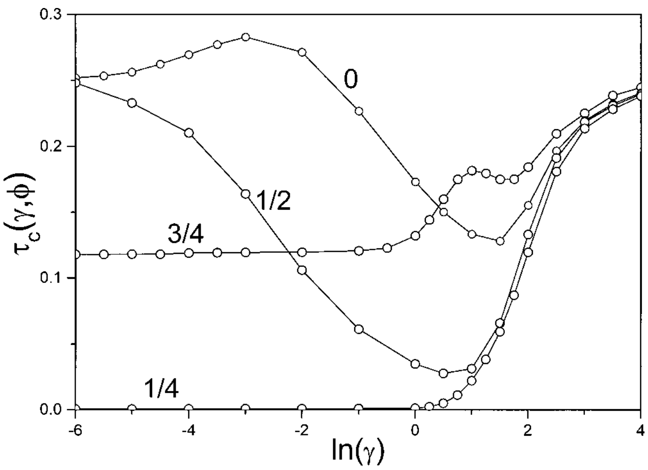

FIG. 4. The relaxation time $\tau_c=\tau_c(\gamma,\phi)$ versus the harmonic oscillations frequency γ at selected values of the phase $\phi=0$, 1/4, 1/2, and 3/4 (as labeled). Reflecting boundary conditions, and $V_a=10$. See text for a discussion of the numerical accuracy of these curves.

$$\tau_c(\gamma,\phi)=\frac{\pi}{2}\int_0^{\infty} dt[b_1(t)-b_1^{(s)}(t)], \tag{15}$$

where $b_1^{(s)}(t)$ is the (computed) periodic stationary limit of the function $b_1(t)$. Within the chosen normalization the free particle results $b_1(t)=(2/\pi)\exp(-4t)$ and $b_1^{(s)}(t)=0$ lead to the limiting value $\tau_c(0,0)=1/4$.

The computed curves $\tau_c(\gamma,\phi)$ are shown in Fig. 4 for $\phi=0$, 1/4, 1/2, and 3/4. The curves are directly comparable to those of Fig. 2, but in the $\phi=3/4$ curve the resonant minimum is superimposed over an increasing function of γ, giving rise to a closely spaced pair of a maximum and a minimum. We also remark that Eq. (15) calls for subtraction of two inexactly known oscillating functions, and that curves $\tau_c(\gamma,\phi)$ may therefore be burdened by a systematic numerical error: There is, correctly, $\lim_{\gamma\to\infty}\tau_c(\gamma,\phi)=1/4$, and $\tau_c(0,\phi)=1/4$ for $\phi=0$ and 1/2, and also $\tau_c(0,3/4)\approx 0.118$ is comparable to the limiting value obtained in a separate calculation for a static potential. A similar calculation, however, yields $\tau_c(0,1/4)\approx 3.24\times 10^{-4}$, which is more than an order of magnitude smaller than the values used here in Fig. 4. Despite these numerical defects we may conclude that a closed system with only intrawell thermal relaxation has the same resonant properties, and the same dependence on initial conditions, as the two models of thermally driven decay studied in Secs. I and II.

[1] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992); C. Van den Broeck, Phys. Rev. E **47**, 4579 (1993); I. Klik and Y. D. Yao, *ibid.* **62**, 4469 (2000).
[2] M. Bier and R. D. Astumian, Phys. Rev. Lett. **71**, 1649 (1993).
[3] P. Reimann, Phys. Rev. Lett. **74**, 4576 (1995).
[4] P. Pechukas and P. Hänggi, Phys. Rev. Lett. **73**, 2772 (1994); A. J. R. Madureíra, P. Hänggi, V. Buonomano, and W. A. Rodrigues, Jr., Phys. Rev. E **51**, 3849 (1995); M. Marchi, F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci, *ibid.* **54**, 3479 (1996).
[5] W. H. Press, B. H. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes* (FORTRAN Version) (Cambridge University Press, Cambridge, 1990), Sec. 15.6.
[6] J. Pachner, *Handbook of Numerical Analysis Applications* (McGraw-Hill, New York, 1984), Sec. 3.3.